the U. S. Atomic Energy Commission (sponsors of the work) cognizant engineer, Mr. Ron Scroggins.

NOTATION

A = heat transfer area, sq. ft.
A_w = wetted area, sq. ft.
Q = heat transfer rate, B.t.u./hr.
R_a = total resistance without boiling, ohms
R_b = resistance across the region of vapor formation with boiling, ohms
R_{nb} = resistance across the region of vapor formation with no boiling, ohms
R_t = total resistance with boiling, ohms
T = temperature, °F.
ΔT = temperature difference (wall minus saturation), °F.
ΔT_o = overall temperature difference (Steam temperature-alcohol boiling point), °F.

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Simplification of the Mathematical Description of Boundary and Initial Value Problems

J. D. HELLUMS
Rice University, Houston, Texas
S. W. CHURCHILL
The University of Michigan, Ann Arbor, Michigan

Methods of reducing the number of parameters and independent variables in a mathematical model have been investigated for many years. Investigations of the parametric description of a problem are carried out under such terminology as dimensional analysis, modeling, scale-up laws, and inspectional analysis. Methods of reducing the number of independent variables are usually considered separately. The method of similarity transformations is discussed by Schlichting (*1*). In this well-known method the transformations of variable are assumed to be of a certain general form and are then defined specifically by direct introduction into the original system of equations. Birkhoff (*2*) describes a different approach based on group theory. In this latter method, called *the method of search for symmetric solutions*, a group of transformations is found under which the differential equations and conditions are invariant. Then a solution is sought which is invariant under the same transformation.

The fundamental simplicity and power of the method of search for symmetric solutions are well known, at least among workers in group theory. However there are two important extensions of this method which do not seem to have been noted or generally appreciated.

One extension is that the problem of finding the minimum parametric description can be directly related to the problem of finding the minimum description in terms of independent variables. In fact any systematic method of inspectional or dimensional analysis which leads to the minimum number of parameters can also be shown to lead directly to similarity transformations in many cases. This consequence makes it possible to find the minimum description of a problem in terms of both variables and parameters by use of a single, direct method. This new direct method is the subject of this paper. The new method is easily shown to be equivalent to the two separate procedures used by Birkhoff: "inspectional analysis" and "search for symmetric solutions."

The second extension is the use of the method of search for symmetric solutions and hence the method described in this paper for problems involving arbitrary functions. The new method may thus be applied to yield the classes of functions which admit the possibility of a similarity transformation. Needless to say a reduction in the number of independent variables constitutes an important simplification whether the reduced problem is to be solved analytically, numerically, or experimentally. Application of the method to arbitrary functions is particularly useful in that the effect of idealizations may be examined in detail. In many cases it is possible to find similar solutions, approximate solutions, or solutions of asymptotic validity.

The method will be illustrated below for several problems of general interest. A simplified form of the method has been discussed in an earlier paper on natural convection (*3*) in which only cases of fixed mathematical description were considered. In the more general method described below it is possible to examine in detail the implications of various assumptions in the model. Attention will be confined to asymptotic solutions which are suggested naturally in the method of analysis. The general problem of asymptotic behavior has been the subject of a literature beyond the scope of this paper; the reader is referred to a review paper by Friedrichs (4) for a particularly interesting survey. Suggestions for further research on the application of group theory to partial differential equations are given by Krzywoblocki (5). It should also be noted that Morgan (6) and, more recently Manohar (7) have discussed the application of group theory to problems involving arbitrary flow distributions outside the boundary layer.

OUTLINE OF THE NEW METHOD

The method of analysis consists of the following steps:

1. The variables, parameters, boundary conditions, and initial conditions are placed in dimensionless form by the introduction of arbitrary reference variables. It is also necessary to place each arbitrary function in dimensionless form by introduction of the function in terms of the reference variables as a reference quantity.
2. Each dimensionless parameter is equated to a constant. This procedure yields a system of algebraic equations in the reference quantities.
3. This set of equations is solved to yield expressions of the reference quantities in terms of the parameters of the original problem. If the system is over determined, it is not possible to eliminate all parameters by choice of the reference quantities, and one parameter will appear in the problem for each algebraic equation which cannot be satisfied.
4. If the system is underdetermined, that is to say if all of the independent algebraic equations may be satisfied without specifying all of the reference quantities, this degree of freedom may be used to reduce the number of independent variables. The dimensionless variables are therefore combined in such a way as to eliminate the remaining arbitrary reference quantities.
5. In problems involving arbitrary functions it is often important to determine what class of functions will admit a reduction in the number of independent variables. This question can be resolved by finding those functions which leave one or more reference quantities arbitrary.

The details of the method are most easily explained by the example following.

ONE-DIMENSIONAL CONDUCTION

As an introduction to the method of analysis consider the familiar problem of heat conduction to a semi-infinite slab with constant boundary and initial conditions:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (1)$$

$$T(0, t) = T_S \quad (2)$$

$$T(y, 0) = T_i \quad (3)$$

The problem is placed in dimensionless form by formally substituting the dimensionless variables $\tau = t/t_o$, $Y = y/y_o$, and $U = (T - T_A)/T_o$ for the original variables T_A, T_o, t_o and y_o are regarded as arbitrary. Additive reference quantities such as T_A can also be introduced for the independent variables, but this generally does not lead to a further simplification in the description of the problem.

Thus

$$\frac{\partial U}{\partial \tau} = \frac{\alpha t_o}{y_o^2} \frac{\partial^2 U}{\partial Y^2} \quad (4)$$

$$U(0, \tau) = (T_S - T_A)/T_o \quad (5)$$

$$U(Y, 0) = (T_i - T_A)/T_o \quad (6)$$

This new statement of the problem contains three parameters: $\alpha t_o/y_o^2$, $(T_i - T_A)/T_o$, and $(T_S - T_A)/T_o$. These can be formally eliminated from the problem by equating the first two to unity and the third to zero. The solution of the resulting set of three equations determines three reference quantities, say $T_A = T_S$, $T_o = T_i - T_S$, and $t_o = y_o^2/\alpha$. The fourth, y_o, remains arbitrary. The problem is now free of parameters, and it is concluded that $(T - T_S)/(T_i - T_S)$ depends only on $t\alpha/y_o^2$ and y/y_o. The original problem is independent of the parameter y_o*. Hence a solution must be sought which is also independent of y_o. Such a solution is found by direct elimination of y_o between the dimensionless variables to be of the form

$$(T - T_S)/(T_i - T_S) = f(t\alpha/y^2) \quad (7)$$

which is of course the well-known transformation that reduces the partial differential equation to an ordinary differential equation.

Consider a variation on the problem in which the constant surface temperature boundary condition, Equation (2), is replaced by a constant flux condition

$$q = k \frac{\partial T(0, t)}{\partial y} \quad (8)$$

where q and k are constants. The same procedure as before this time yields the parameter-free result

$$(T - T_i) \frac{k}{(q\sqrt{\alpha t})} = j\ (t\alpha/y^2) \quad (9)$$

The indicated transformation again yields an easily soluble ordinary differential equation and boundary conditions. Comparison of Equations (7) and (9) indicates that the boundary conditions have an important role in determining the variables which yield the simplest description of the problem.

FORCED AND FREE CONVECTION IN A BOUNDARY LAYER

As a second illustration consider the problem of two-dimensional boundary layer flow involving both forced and free convection. At the outset the velocity outside the boundary layer, the pressure gradient, and the orientation and shape of the surface will be regarded as arbitrary. Then particular special cases will be determined for which a simplified description of the problem is possible. With the usual assumptions of boundary layer theory the conservation equations may be written as follows:

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = f_1(x) + f_2(x)\phi + \nu \frac{\partial^2 u}{\partial y^2} \quad (10)$$

$$u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = \alpha \frac{\partial^2 \phi}{\partial y^2} \quad (11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (12)$$

and the boundary conditions as

$$u = h_3(x) \text{ and } \phi = 0 \text{ at } y = \infty \quad (13)$$

$$u = h_3(0) \text{ and } \phi = 0 \text{ at } x = 0 \quad (14)$$

$$u = v = 0 \text{ and } \phi = 1 \text{ at } y = 0 \quad (15)$$

where $\phi = (T - T_\infty)/(T_S - T_\infty)$, $f_1(x)$ corresponds to $-\frac{1}{\rho}\frac{dP}{dx} - g \sin\theta$, $f_2(x)$ corresponds to $(g\beta\Delta T) \sin\theta$, and θ is the angle of inclination of the surface from the hori-

* In the notation of Birkhoff (2) the problem has been shown to be invariant under the transformations $t \to at$, $x \to \sqrt{a}y$, and $T \to T$.

zontal. For the sake of simplicity the reference quantities for temperature were chosen in advance, based on experience.

The equations are placed in dimensionless form by letting $U = u/u_o$, $V = v/v_o$, $X = x/x_o$, $Y = y/y_o$, $F_1 = f_1(x)/f_1(x_o)$, $F_2 = f_2(x)/f_2(x_o)$, and $H_3 = h_3(x)/h_3(x_o)$. The dimensionless parameters which result are $u_o y_o^2/\nu x_o$, $v_o y_o/\nu_o$, $f_1(x_o) y_o^2/\nu u_o$, $f_2(x_o) y_o^2/\nu u_o$, $u_o y_o^2/\alpha x_o$, $v_o y_o/\alpha$, and $u_o/h_3(x_o)$.

From inspection or consideration of the algebraic problem resulting from equating each group to unity it is found that four of the six independent groups can be reduced to unity by the following choice of the reference quantities: $u_o = [\alpha x_o f_1(x_o)/\nu]^{1/2}$ $v_o = [\alpha^3 f_1(x_o)/\nu\, x_o]^{1/4}$ and $y_o = [\alpha\nu x_o/f_1(x_o)]^{1/4}$, with x_o still regarded as arbitrary.

The differential equations in terms of the new variables are

$$\frac{\alpha}{\nu} \quad U\frac{\partial U}{\partial Y} + V\frac{\partial U}{\partial Y} = \frac{f_2(x_o)}{f_1(x_o)}\phi F_2 + F_1 + \frac{\partial^2 U}{\partial Y^2} \tag{16}$$

$$U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y} = \frac{\partial^2 \phi}{\partial Y^2} \tag{17}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{18}$$

and the boundary condition on velocity is

$$U = h_3(x_o)\left[\frac{\nu}{\alpha x_o f_1(x_o)}\right]^{1/2} H_3 \text{ at } Y = \infty \tag{19}$$

A reduction in the number of independent variables (a similarity transformation) may be possible since the parameter x_o is arbitrary. However for such a transformation to exist it is necessary that the parameters and functions of the problem be independent of x_o. Several cases where similarity transformations are possible are discussed below.

Forced Convection to Plane Surfaces

In this case $F_2 = f_2 = 0$, and θ is constant. Then $u[\nu/\alpha x_o f_1]^{1/2}$, $u[\nu x_o/\alpha^3 f_1]^{1/4}$, and ϕ depend only on the variables $y[f_1/\alpha\nu x_o]^{1/4}$ and x/x_o, the parameters α/ν and $h_3(x_o)[\nu/\alpha x_o f_1(x_o)]^{1/2}$, and the functions F_1 and H_3.

The usual procedure in such problems is either to regard the pressure distribution as a specified function of x or as related to the flow at the edge of the boundary layer by the relation $f_1 = h_3\left(\frac{dh_3}{dx}\right)$. Substituting this relation into the boundary condition on U one obtains

$$U = h_3(x_o)\left[\frac{\alpha x_o h_3(x_o)}{\nu}\frac{dh_3(x_o)}{dx}\right]^{-1/2} H_3 \text{ at } Y = \infty \tag{20}$$

This coefficient of H_3 as well as the functions and composite variables must be independent of x_o for a similar solution to exist. The coefficient may therefore be equated to a constant. Solution of the resulting equation gives the class of functions h_3 for which similarity transformations are possible. The requirement is that $h_3 = cx^n$, which is the well-known result (*1*) corresponding to stagnation or wedge flows depending on the value of the constant n. $n = 0$ corresponds to the particular case of parallel flow over a flat plate. Since in this case $f_1(x) = 0$ and h_3 is constant, $f_1(x_o)$ may be chosen to eliminate the parameter in the boundary conditions:

$$f_1(x_o) = (h_3^2\nu)/(\alpha x_o) \tag{21}$$

This choice yields $y_o = (\alpha x_o/h_3)^{1/2}$ with α/ν the only parameter remaining in the problem. The solution for the temperature is thus of the form

$$\phi = G[y(h_3/\alpha x)^{1/2}, \alpha/\nu] \tag{22}$$

which may be differentiated and rearranged in the form

$$N_{Nu} = N_{Re}^{1/2}\ N_{Pr}^{1/2}\ E(N_{Pr}) \tag{23}$$

It is interesting to consider the asymptotic solution for large N_{Pr}. The form of the equations shows that the inertial terms become relatively less important as the Prandtl number increases. However it is not possible simply to neglect the inertial terms in the limit and regard the function as approaching a constant. This conclusion is evident from the fact that the momentum balance is independent of the energy balance in the simplified system of equations.

To find the form of the asymptotic solution it is necessary to note that u/h_3 depends only on $y(h_3/\nu x)^{1/2}$ and that large Prandtl numbers correspond to small values of the argument. The first term of a series expansion about zero is $u/h_3 = Py(h_3/\nu x)^{1/2}$, where P is constant. Substitution of this expression into the continuity equation, integration, substitution into the energy balance, and a reapplication of the method of analysis leads directly to the asymptotic expression

$$N_{Nu} = B\ N_{Re}^{1/2}\ N_{Pr}^{1/3} \tag{24}$$

where B is a constant.

This asymptotic expression (in a more general form) was apparently first developed by Lighthill (*8*) many years after Polhausen (*9*) had reported the same expression as an interpolation formula for his results. Acrivos (*10*) has developed the corresponding relationship for non-Newtonian, power-law fluids.

Free Convection to Plane Surfaces

This case, for which $F_1 = 0$, $F_2 = 1$, $H_3 = 0$ and $f_2 = (g\beta\Delta T)\sin\theta$, is discussed for a variety of cases including unsteady state in reference 3.

Forced and Free Convection

For the more general case where both forced and free convection are important it is evident from inspection of the dimensionless equations that for the parameter x_o to be arbitrary it is necessary that f_1 and f_2 be constants and that $h_3 = cx^{1/2}$. This corresponds to flow under a constant pressure gradient and is a restricted case of the wedge-flow problem discussed above. Hence it is possible to reduce the more general problem to one in ordinary differential equations, but only for a restricted set of boundary conditions.

Forced Convection with Interphase Mass Transfer

Important variations on the boundary-layer problem occur in situations where mass transfer (or boundary-layer suction) induces a normal component of the velocity at an interface. Consider for example the classical problem of flow along a flat surface with a normal component of the velocity at the surface of $g(x)$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{26}$$

$$u = 0 \text{ and } v = g(x) \text{ at } y = 0 \tag{27}$$

$$u = u_\infty \text{ at } x = 0 \text{ and } y = \infty \tag{28}$$

Two important questions involving $g(x)$ are: for what functions $g(x)$ is it possible to find a similarity transformation, and what is the asymptotic solution for large $g(x)$.

The first question is easily answered by the procedure outlined above: $g(x) = Ax^{-1/2}$ where A is a constant. This form of the function is consistent with the boundary layer diffusion equation, and A can be expressed in terms of the mass transfer parameters of the problem. [See the next section of this paper and reference *11*, p. 608 f.]

The asymptotic solution requires a slightly different treatment of the variables than that given in the previous examples. The variable v appears in the problem in terms linear in v. In such cases, as is usual in energy balance problems, a linear combination of variables often yields a simpler description of the problem than merely a ratio of variables. In the problem at hand the choice of variable $V = (v - Ax^{-1/2})/v_o$ removes the function from boundary conditions and permits the choice of v_o to eliminate a parameter from the differential equations. Carrying out the procedure as before one obtains the one-parameter problem:

$$\left[\frac{u_\infty \nu}{A^2}\right] \left[U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right] + \frac{1}{X^{1/2}}\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (30)$$

with

$$U = V = 0 \text{ at } Y = 0 \text{ and } U = 1 \text{ at } X = 0 \text{ and at } Y = \infty \qquad (31)$$

where

$Y = yA/\nu x_o^{1/2}$, $X = x/x_o$, $U = u/u_\infty$, and $V = (v - Ax^{-1/2})\ (Ax_o^{1/2}/u_\infty \nu)$

with x_o arbitrary. [There is some freedom in the choice of dimensionless variables which reduce the problem to a single parameter. However there is a unique set of variables which, in the limit of large A, yield a problem consistent with the boundary conditions.]

Using the freedom in the reference quantity to reduce the number of independent variables, and neglecting the first two inertial terms for large A, one gets by direct integration

$$u/u_\infty = 1 - \exp\ (yA/\nu x^{1/2}) \qquad (32)$$

which is easily shown to be a good approximation to the solution of the original problem over a wide range of the parameter [see (*1*) for example]. It is evident that Equation (32) can satisfy the condition $u \to u_\infty$ as y increases only if A is negative. Hence this asymptotic solution is restricted to the case of mass transfer toward the interface.

LAMINAR CONDENSATION WITH NONCONDENSABLE GASES

The previous sections of this paper have been concerned with the application of the method of analysis to problems of classical interest. In this section the application of the method to a problem of current interest will be illustrated. Sparrow and Eckert (*12*) recently presented an analysis of condensation on vertical flat surfaces which confirmed a previous qualitative conclusion that noncondensables severely reduce the rate of condensation. However they concluded that their results were inadequate from a quantitative viewpoint because the effect of natural convection was neglected in the analysis. It will be shown below that a model for this problem including natural convection effects can be reduced by a similarity transformation.

Several recent papers (*12, 13, 14*) improve on Nusselt's original solution for condensation of a pure component. However it is clear that the liquid film behavior is less important with noncondensables than in the pure component case. In many practical problems the temperature drop occurs almost entirely in the vapor phase, so that for a first approximation it is possible to assume both that the liquid film follows Nusselt's flow relation and that the gas-liquid interface is isothermal.

Under these conditions the liquid and vapor flow problems are related in a simple way. If $u_S(x)$ is the velocity of the interface, it is related to the liquid mass flow rate by the following relation derived in reference *11*:

$$u_S = [9g\Gamma^2/8\mu_L\rho_L]^{1/3} \qquad (33)$$

where Γ the liquid mass flow rate per unit width is related to the component of the vapor velocity normal to the interface as

$$\Gamma = \int_0^x \rho_S v_S \, dx \qquad (34)$$

with

$$v_S = \frac{-D\Delta W}{1 - W_S}\left(\frac{\partial\psi}{\partial y}\right)_{y=0} \qquad (35)$$

It is then possible to treat the liquid flow as a boundary condition for the vapor. The complete boundary conditions are

$$u = u_S = M\left[\int_0^x \frac{\partial\psi}{\partial y}\,dx\right]^{2/3},$$

$$v = \frac{-D\Delta W}{1 - W_S}\frac{\partial\psi}{\partial y}, \text{ and } \psi = \phi = 1 \text{ at } y = 0 \qquad (36)$$

$$u = \psi = \phi = 0 \text{ at } x = 0 \text{ and at } y = \infty \qquad (37)$$

where

$$M = [(9g/8\mu_L\rho_L)(\rho D\Delta W/[1 - W_S])^2]^{1/3} \qquad (38)$$

The differential equations for the vapor are given below. These equations are of the boundary layer type with the assumption of a constant property fluid except in the buoyant-force terms.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\,\phi\Delta T + g\gamma\psi\Delta W + \nu\frac{\partial^2 u}{\partial y^2} \qquad (39)$$

$$u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} = \alpha\frac{\partial^2\phi}{\partial y^2} \qquad (40)$$

$$u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} = D\frac{\partial^2\psi}{\partial y^2} \qquad (41)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (42)$$

The dimensionless groups which appear in the momentum balance after it is placed in dimensionless form are $u_o y_o^2/\nu x_o$, $v_o y_o/\nu$, $g\beta y_o^2\Delta T/\nu u_o$, and $g\gamma y_o^2\Delta W/\nu u_o$. The energy and diffusion equations contain dimensionless groups corresponding to the first two listed above with ν replaced by α and D, respectively. The groups which appear in the boundary conditions are $D\Delta W/v_o y_o(1 - W_S)$ and $Mx_o^{2/3}/u_o y_o^{2/3}$. The algebraic procedure outlined before yields the result that one of the reference quantities is arbitrary and that three of the groups may be eliminated by choosing

$y_o = [\nu D x_o/g\gamma\Delta W]^{1/4}$, $u_o = (x_o D g\gamma\Delta W/\nu)^{1/2}$, and $v_o = [D^3 g\gamma\Delta W/\nu x_o]^{1/4}$

The problem in terms of the dimensionless variables is then

$$\frac{D}{\nu}\left[U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right] = \frac{\beta\Delta T}{\gamma\Delta W}\phi + \psi + \frac{\partial^2 U}{\partial Y^2} \qquad (43)$$

$$\frac{D}{\alpha}\left[U\frac{\partial\phi}{\partial X}+V\frac{\partial\phi}{\partial Y}\right]=\frac{\partial^2\phi}{\partial Y^2} \quad (44)$$

$$U\frac{\partial\psi}{\partial X}+V\frac{\partial\psi}{\partial Y}=\frac{\partial^2\psi}{\partial Y^2} \quad (45)$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \quad (46)$$

and the boundary conditions are

$$U=M'\left[\int_o^x\frac{\partial\psi}{\partial Y}dX\right]^{2/3},$$

$$V=\frac{-\Delta W}{1-W_S}\frac{\partial\psi}{\partial Y}, \text{ and } \psi=\phi=1 \text{ at } Y=0 \quad (47)$$

$$U=\psi=\phi=0 \text{ at } X=0 \text{ and at } Y=\infty \quad (48)$$

where

$$M'=M(\nu/g\gamma D^2\Delta W)^{1/3} \quad (49)$$

At this point it has been established that the variables $u(\nu/xDg\gamma\Delta W)^{1/2}$, $v(\nu x/D^3g\gamma\Delta W)^{1/4}$, and ϕ depend only on $y(g\gamma\Delta W/\nu Dx)^{1/4}$ in addition to the parameters. This constitutes an important simplification of the problem, since a system of ordinary differential equations may be developed and solved numerically. It is also interesting to consider certain limiting cases.

It is evident that when $\beta\Delta T << \gamma\Delta W$, the momentum balance and the diffusion equation are independent of the energy balance. In the still more limited case of large Schmidt numbers, where the inertial terms in the momentum balance are neglected, the solution reduces to the form

$$N'_{Nu}=(N'_{Gr}N_{Sc})^{1/4}\cdot J[M',\Delta W/(1-W_s)] \quad (50)$$

Further simplification may be possible over certain ranges of the parameters.

DISCUSSION

The examples given indicate the utility of the method. It is of course clear that similarity transformations exist for only a small fraction of problems of practical interest. In such cases the method described here will yield the minimum parametric description of the model and may be used to suggest a modified model which would admit a transformation or other simplification. The same results can in principle always be found by other methods, but the method given here seems to be the simplest and most direct. The boundary layer examples were chosen as examples of some familiarity for the purpose of illustration. The condensation example was chosen to illustrate the application of the method to a problem with more complex boundary conditions. This problem is of considerable practical importance and apparently has not been analyzed before.

NOTATION

A = constant
a = constant
B = constant
C = constant
D = diffusion coefficient
$E(x)$ = function of x
$f(x)$ = function of x
$G(x)$ = function of x
g = acceleration due to gravity
$g(x)$ = function of x
$h(x)$ = function of x
$J(x)$ = function of x
$j(x)$ = function of x
k = thermal conductivity
M = group of variables defined by Equation (38)
M' = group of variables defined by Equation (49)
N_{Nu} = local Nusselt number = $-x(\partial\phi/\partial y)_{y=0}$
N'_{Nu} = local Nusselt number for mass transfer = $-x(\partial\psi/\partial y)_{y=0}$
N'_{Gr} = Grashof number for mass transfer, $x^3g\gamma\Delta W/\nu^2$
N_{Pr} = Prandtl number = ν/α
N_{Re} = Reynolds number = $u_\infty x/\nu$
N_{Sc} = Schmidt number = ν/α
P = pressure
q = heat flux density
T = temperature
t = time
u = velocity parallel to interface
v = velocity normal to interface
W = weight fraction of diffusing component
x = distance along surface from leading edge
y = distance normal to surface

Greek Letters

α = thermal diffusivity
β = coefficient of expansion $\approx \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_W$
r = liquid mass flow rate per unit width
γ = coefficient of expansion $\approx \frac{1}{\rho}\left(\frac{\partial\rho}{\partial W}\right)_T$
μ = viscosity
ν = kinematic viscosity
ϕ = $(T-T_\infty)/(Ts-T_\infty) = (T-T_\infty)/(\Delta T)$
θ = angle of inclination of surface to horizontal
ρ = density
ψ = $(W-W_\infty)/(Ws-W_\infty) = (W-W_\infty)/(\Delta W)$
Γ = liquid mass flow rate per unit width

Subscripts

A = arbitrary reference quantity
S = at the interface
∞ = in the bulk fluid
o = arbitrary reference quantity
L = liquid
i = initial

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